Analysis III Mid-semestral Examination, 2008, B.Math 2nd Year

Attempt all questions. Each question carries 10 marks. You may consult books and notes, and cite results proved in class without re-proving them. Results of exercises, however, must be proved if cited.

1. (i): Let $x = (a, b) \in \mathbb{R}^2$, and let $f : U \to \mathbb{R}$ be a map of a neighbourhood U of x which is differentiable at x. Compute the limit:

$$\lim_{h \to 0} \frac{f(a + h, b + \sin h) - f(a, b)}{h}$$

in terms of the two partial derivatives of f at x.

(ii): Let $n \geq 2$ and $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^n$ be a C^1 map satisfying $\|f(x)\| = 1$ for all $x \in U$. Show that f is not a submersion at any point of U.

2. (i): Let $n \geq 2$ and $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function satisfying $\frac{\partial f}{\partial x_1}(x) \equiv 0$ for all $x \in \mathbb{R}^n$. Show that there exists a C^1 function $g : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $f(x_1, .., x_n) = g(x_2, ..., x_n)$ for all $(x_1, .., x_n) \in \mathbb{R}^n$ (viz. f is independent of x_1).

(ii): Let $n \geq 2$ and $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. Show that f is not injective.

3. (i): Let $f : [a, b] \to \mathbb{R}$ be a bounded monotonically non-decreasing function (i.e. $x < y \Rightarrow f(x) \leq f(y)$). If $x_1, x_2, .., x_n$ are distinct points in $[a, b]$, show that

$$\sum_{i=1}^{n} o(f, x_i) \leq f(b) - f(a)$$

where $o(f, x_i)$ is the oscillation of f at x_i.

(ii): Let $f : [a, b] \to \mathbb{R}$ be as in (i) above. Show that f is integrable.

4. (i): Let A be a closed subset of $\mathbb{R}^n$ and let U be an open subset of $\mathbb{R}^n$ with $A \subset U$. Show that there exists a C^∞ function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying (i) $f(x) \equiv 1$ for all $x \in A$ and (ii) $f(x) \equiv 0$ for all $x \in U^c$.

(ii): Compute the area of the planar region:

$$A = \{(x, y) \in \mathbb{R}^2 : 1 \leq xy \leq 2, \ x \leq y \leq 2x\}$$